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A Plancherel measure associated to set partitions and its limit

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Abstract. In recent years increasing attention has been paid on the area of supercharacter theories, especially to those of the upper unitriangular group. A particular supercharacter theory, in which supercharacters are indexed by set partitions, has several interesting properties, which make it object of further study. We define a natural generalization of the Plancherel measure, called superplancherel measure, and prove a limit shape result for a random set partition according to this distribution. We also give a description of the asymptotic behavior of two set partition statistics related to the supercharacters. The study of these statistics when the set partitions are uniformly distributed has been done by Chern, Diaconis, Kane and Rhoades.

Keywords: supercharacter, set partition, Plancherel measure.

1 Introduction

Let p be a prime number, q a power of p, and $\mathbb{K}$ the finite field of order q and characteristic p. Consider $U_n = U_n(\mathbb{K})$, the group of upper unitriangular matrices with entries in $\mathbb{K}$, it is known that the description of conjugacy classes and irreducible characters of U_n is a wild problem. To bypass the issue, André [2] and Yan [15] set the foundations of what is now known as "supercharacter theory". The idea is to meld together some irreducible characters and conjugacy classes (called respectively supercharacters and superclasses), in order to have characters which are easy enough to be tractable but still carry information of the group. For example, in [3], Arias-Castro, Diaconis and Stanley described random walks on U_n utilizing only the supercharacter table (usually the complete character table is required). More recently, Diaconis and Isaacs [8] formalized the axioms of supercharacter theory, generalizing the construction from U_n to algebra groups.

Of the various supercharacter theories for U_n a particular nice one, hinted in [1] and described by Bergeron and Thiem in [5], has the property that the supercharacters take integer values on superclasses. This is particularly interesting because of a

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result of Keller [10], who proves that for each group G there exists a unique finest supercharacter theory with integer values. Although it is not yet known if Bergeron and Thiem's theory is the finest integral one, it has remarkable properties which make it worth of a deeper analysis. In this theory the supercharacters of U_n are indexed by set partitions of $\{1,\ldots,n\}$ and they form a basis for the Hopf algebra of superclass functions. This Hopf algebra is isomorphic to the Hopf algebra of symmetric functions in noncommuting variables. In this supercharacter theory the characters depend on three statistics defined for a set partition π of $[n]$: $d(\pi)$, which is the number of arcs of π; $\dim(\pi) = \sum \max(B) - \min(B)$, where the sum runs through the blocks B of π; and $\operatorname{crs}(\pi)$, which is the number of crossings of π. More precisely, if χ^π is the supercharacter associated to the set partition π, then $\chi^\pi(1) = q^{\dim(\pi)-d(\pi)}$ and $\langle \chi^\pi, \chi^\pi \rangle = q^{\operatorname{crs}(\pi)}$.

In the setting of probabilistic group theory one is interested in the study of statistics of the "typical" irreducible representation of the group. A natural probability distribution is the uniform distribution; in [7] and [6], Chern, Diaconis, Kane and Rhoades study the statistics dim and crs for a uniform random set partition, proving formulas for the moments of $\dim(\pi)$ and $\operatorname{crs}(\pi)$ and, successively, a central limit theorem for these two statistics.

In representation theory another natural distribution is the Plancherel measure, which is a discrete probability measure associated to the irreducible characters of a finite group. The Plancherel measure has received vast coverage in the literature, especially in the case of the symmetric group S_n, for which a limit shape result was proven, independently, by Kerov and Vershik [12] and Logan and Shepp [13].

From the study of the Plancherel measure of S_n has followed a theory regarding the Plancherel growth process. Indeed, there exist natural transition measures between the partitions of n and the partitions of $n+1$, which generate a Markov process whose marginals are the Plancherel distributions. The transition measures have a nice combinatorial description, see [11].

In this paper we generalize the notion of Plancherel measure to adapt it to supercharacter theories. We call the measure associated to a supercharacter theory *superplancherel measure*. In Section 2.2 we show that for a tower of groups $\{1\} = G_0 \subseteq G_1 \subseteq \ldots$, each group endowed with a "coherent" supercharacter theory, there exists a nontrivial transition measure which yields a Markov process; the marginals of this process are the superplancherel measures. In order to show this, we generalize a construction of superinduction for algebra groups to general finite groups. Such a construction was introduced by Diaconis and Isaacs in [8] and developed by Marberg and Thiem in [14].

We then consider the superplancherel measure associated to the supercharacter theory of U_n described by Bergeron and Thiem. In this setting, the superplancherel measure has an explicit formula depending on the statistics $\dim(\pi)$ and $\operatorname{crs}(\pi)$; we give a direct combinatorial interpretation of this measure.

The main result of the paper is a limit shape for a random superplancherel distributed

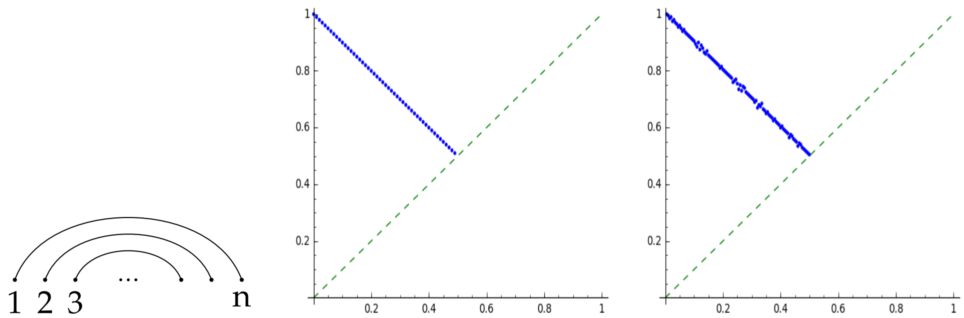


Figure 1: Description of a random superplancherel distributed set partition: the left image is the diagram of the set partition with the highest probability; in the center there is the corresponding measure μ_π; on the right there is a computer generated measure μ_π for $\pi \vdash [200]$. The algorithm we use for the program that generates a big random set partition is based on the combinatorial interpretation in Section 3.1.

set partition. In order to formulate this result we immerse set partitions into the space of subprobabilities (*i.e.*, measures with total weight less than or equal to 1) of the unit square $[0,1]^2$ with some other properties. This embedding is similar to that of permutons for random permutations and graphons for random graphs, see for example [9]. Given a set partition π we refer to the corresponding subprobability as μ_π. We describe a measure Ω such that

$$\mu_\pi \to \Omega \quad \text{almost surely.} \tag{1.1}$$

The measure Ω is the uniform measure on the set $\{(x, 1-x) \text{ s.t. } x \in [0, 1/2]\}$ of total weight $1/2$. Informally, we can say that a set partition chosen at random with the superplancherel measure is asymptotically close to the one with highest probability (see Section 1). In the process, we obtain asymptotic results for the statistics $\dim(\pi)$ and $\operatorname{crs}(\pi)$ when π is chosen at random with the superplancherel measure. Namely, we show that almost surely

$$\dim(\pi) \sim \frac{1}{4}n^2, \qquad \operatorname{crs}(\pi) \in o(n^2). \tag{1.2}$$

Sections 4 and 5 present the main arguments of the proofs of (1.1) and (1.2). As mentioned, the main idea is to consider set partitions as particular measures of the unit square. With this transformation the statistics $\dim(\pi)$ and $\operatorname{crs}(\pi)$ can be seen as integrals of the measure μ_π. We use an entropy argument to delimitate a set of set partitions of maximal probability. Finally, we relate the results on the entropy into the weak* topology of measures of $[0,1]^2$.

2 Preliminaries

2.1 Supercharacter theory and the superplancherel measure

In order to recall the definition of supercharacter theory, we fix some notation of character theory: given a group G we call $\mathrm{Irr}(G)$ the set of irreducible characters. The regular character is

$$\rho_G(g) = \sum_{\xi \in \mathrm{Irr}(G)} \xi(1_G)\xi(g) = \begin{cases} |G| & \text{if } g = 1_G \\ 0 & \text{otherwise,} \end{cases}$$

and the Plancherel measure associated to G is $\mathrm{Pl}_G(\xi) := \xi(1)^2/|G|$. If χ, ξ are characters of G and ξ is irreducible we say that ξ is a *constituent* of χ if $\langle \chi, \xi \rangle \neq 0$, where $\langle \cdot, \cdot \rangle$ is the Frobenius inner product. Moreover, we call $I(\chi) := \{\xi \in \mathrm{Irr}(G) \text{ s.t. } \langle \chi, \xi \rangle \neq 0\}$.

Definition 2.1. *A* supercharacter theory *of a finite group G is a pair $(\mathrm{scl}(G), \mathrm{sch}(G))$ where $\mathrm{scl}(G)$ is a set partition of G and $\mathrm{sch}(G)$ an orthogonal set of nonzero characters of G (not necessarily irreducible) such that:*

1. $|\,\mathrm{scl}(G)| = |\,\mathrm{sch}(G)|$;
2. *every character $\chi \in \mathrm{sch}(G)$ takes a constant value on each member $\mathcal{K} \in \mathrm{scl}(G)$;*
3. *each irreducible character of G is a constituent of one, and only one, of the characters $\chi \in \mathrm{sch}(G)$.*

The elements $\mathcal{K} \in \mathrm{scl}(G)$ are called *superclasses*, while the characters $\chi \in \mathrm{sch}(G)$ are *supercharacters*. It is easy to see that every element $\mathcal{K} \in \mathrm{scl}(G)$ is always a union of conjugacy classes. Since a supercharacter $\chi \in \mathrm{sch}(G)$ is always constant on superclasses we will sometimes write $\chi(\mathcal{K})$ instead of $\chi(g)$, where $\mathcal{K} \in \mathrm{scl}(G)$ is a superclass and $g \in \mathcal{K}$. Notice that irreducible character theory is a supercharacter theory.

Definition 2.2. *Fix a supercharacter theory $T = (\mathrm{scl}(G), \mathrm{sch}(G))$ of G, we define the* superplancherel measure SPl_G^T *of T as follows: given $\chi \in \mathrm{sch}(G)$, then* $\mathrm{SPl}_G^T(\chi) := \frac{1}{|G|}\frac{\chi(1)^2}{\langle \chi, \chi \rangle}$.

Notice that if T is the irreducible character theory, then the superplancherel measure is equal to the usual Plancherel measure. We stress out that the definition of superplancherel measure depends on the supercharacter theory but we will omit it if it is clear from the context.

It is relatively easy to show that this measure is indeed a probability measure. We let the reader work out the details, which will anyway be available in the extended version of this paper.

2.2 Superinduction and transition measure

In this section we extend the notion of superinduction, defined by Diaconis and Isaacs in [8] for algebra groups, to general finite groups, and we use it to define a transition measure. Let G be a finite group, $H \leq G$ a subgroup and $(\mathrm{scl}(G), \mathrm{sch}(G))$ a supercharacter theory for G. Let $\phi\colon H \to \mathbb{C}$ be any function, we set $\phi^0\colon G \to \mathbb{C}$ to be $\phi^0(g) = \phi(g)$ if $g \in H$ and $\phi^0(g) = 0$ otherwise. We set

$$\mathrm{SInd}_H^G(\phi)(g) := \frac{|G|}{|H| \cdot |[g]|} \sum_{k \in [g]} \phi^0(k),$$

where $[g] \in \mathrm{scl}(G)$ is the superclass containing g. By construction, $\mathrm{SInd}_H^G(\phi)$ is a superclass function. A supercharacter version of the Frobenius reciprocity holds: if ψ is a superclass function then

$$\langle \mathrm{SInd}_H^G(\phi), \psi \rangle = \langle \phi, \mathrm{Res}_H^G(\psi) \rangle,$$

where $\mathrm{Res}_H^G(\psi)$ is the restriction of ψ to H. The proof of this equality is elementary, and we leave the details to the reader.

Consider now also H endowed with a supercharacter theory $(\mathrm{scl}(H), \mathrm{sch}(H))$. Suppose also that this supercharacter theory is *coherent* with the one of G, that is, for each $\mathcal{H} \in \mathrm{scl}(H)$ there exists $\mathcal{K} \in \mathrm{scl}(G)$ such that $\mathcal{H} \subseteq \mathcal{K}$. Notice that this is equivalent to the requirement that $\mathrm{Res}_H^G(\chi)$ is a superclass function on H for each $\chi \in \mathrm{sch}(G)$ by [8, Theorem 2.2].

Definition 2.3. *Let $\chi \in \mathrm{sch}(G)$, $\gamma \in \mathrm{sch}(H)$. The* transition measure *$\rho_H^G(\gamma, \chi)$ is defined as*

$$\rho_H^G(\gamma, \chi) := \frac{|H|}{|G|} \frac{\chi(1)}{\gamma(1)} \frac{\langle \mathrm{SInd}_H^G(\gamma), \chi \rangle}{\langle \chi, \chi \rangle}.$$

Proposition 2.4. *The following hold:*

1. *For each $\chi \in \mathrm{sch}(G)$ we have $\sum_{\gamma \in \mathrm{sch}(H)} \rho_H^G(\gamma, \chi)\, \mathrm{SPl}_H(\gamma) = \mathrm{SPl}_G(\chi)$.*

2. *For each $\gamma \in \mathrm{sch}(H)$ we have $\sum_{\chi \in \mathrm{sch}(G)} \rho_H^G(\gamma, \chi) = 1$.*

In particular, let $\{1\} = G_0 \hookrightarrow \ldots \hookrightarrow G_n \hookrightarrow G_{n+1} \hookrightarrow \ldots$ be a tower of groups, and suppose that for each n we associate a supercharacter theory T_n to G_n which is coherent with T_{n+1}. Let χ_1 be the unique supercharacter for $G_0 = \{1\}$; consider the Markov process with initial state χ_1 and transition measures $\rho_{G_n}^{G_{n+1}}$. Then this process has marginals distributed as SPl_n.

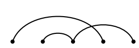
We omit the proof of this proposition, which is straightforward from the definitions.

3 Supercharacter theory for unitriangular matrices

Let $\mathbb{K}$ be the finite field of order q and characteristic p. The group $U_n = U_n(\mathbb{K})$ is the group of upper unitriangular matrices of size $n \times n$ and entries belonging to $\mathbb{K}$, that is,

$$U_n = U_n(\mathbb{K}) = \left\{ \begin{bmatrix} 1 & a_{1,2} & \cdots & a_{1,n} \\ & 1 & a_{2,3} & \vdots \\ & & \ddots & a_{n-1,n} \\ & & & 1 \end{bmatrix} \in M_{n\times n}(\mathbb{K}) \right\}.$$

In [5], Bergeron and Thiem describe a supercharacter theory in which both $\mathrm{sch}(U_n)$ and $\mathrm{scl}(U_n)$ are in bijection with sets partitions of $[n] = \{1, \ldots, n\}$. Through the section, given set partitions $\pi, \sigma \vdash [n]$ we will write χ^π for the supercharacter corresponding to π and $\mathcal{K}_\sigma$ for the superclass corresponding to σ.

This supercharacter theory has an explicit formula for the supercharacter values; in order to recall it we need to set some notation: fix n and a set partition of $\pi \vdash [n]$, if two numbers i and j are in the same block of the set partition $\pi \vdash [n]$ and there is no k in that block such that $i < k < j$, then the pair (i, j) is said to be an *arc* of π. The set partition π is uniquely determined by the set $D(\pi)$ of arcs. The *dimension* $\dim(\pi)$ is $\dim(\pi) := \sum_{(i,j)\in D(\pi)} j - i$; the number of crossings of π is $\mathrm{crs}(\pi)$ of π, where a *crossing* is an unordered pair of arcs $\{(i, j), (k, l)\} \subseteq D(\pi)$ such that $i < k < j < l$; the number of nestings is $\mathrm{nst}(\pi)$, where a *nesting* is an unordered pair of arcs $\{(i, j), (k, l)\} \subseteq D(\pi)$ such that $i < k < l < j$. Given i, j with $i < j \leq n$, we say that the pair (i, j) is π-regular if there exists no $k < i$ such that $(k, j) \in D(\pi)$ and there exists no $l > j$ such that $(i, l) \in D(\pi)$. The set of π-regular pairs is denoted $\mathrm{Reg}(\pi)$. For example, if $\pi = \{\{1, 4\}, \{2, 3, 5\}\} =$  , then $\mathrm{Reg}(\pi) = \{(1, 4), (1, 5), (2, 3), (2, 5), (3, 5)\}$; if an arc is not regular then it is called *singular* and the set of π-singular pairs is denoted $\mathrm{Sing}(\pi)$. In the previous example thus $\mathrm{Sing}(\pi) = \{(1, 2), (1, 3), (2, 4), (3, 4), (4, 5)\}$. By a counting argument one can prove that the cardinality $|\mathrm{Sing}(\pi)|$ is

$$\begin{aligned} |\mathrm{Sing}(\pi)| = \sum_{(i,j)\in D(\pi)} 2(j - i - 1) - \sharp\{(i, j), (l, k) \in D(\pi) \text{ s.t. } i < l < j < k\} \\ = 2(\dim(\pi) - d(\pi)) - \mathrm{crs}(\pi). \end{aligned}$$

Given $\pi, \sigma \vdash [n]$, a formula for $\chi^\pi(\mathcal{K}_\sigma)$ is proven in [4]. In particular the formula implies that

$$\chi^\pi(1) = (q - 1)^{d(\pi)} \cdot q^{\dim(\pi) - d(\pi)} \qquad \text{and} \qquad \langle \chi^\pi, \chi^\pi \rangle = (q - 1)^{d(\pi)} q^{\mathrm{crs}(\pi)};$$

hence we obtain

$$\mathrm{SPl}_n(\chi^\pi) := \mathrm{SPl}_{U_n}(\chi^\pi) = \frac{1}{q^{\frac{n(n-1)}{2}}} \frac{(q - 1)^{d(\pi)} \cdot q^{2\dim(\pi) - 2d(\pi)}}{q^{\mathrm{crs}(\pi)}}.$$

$$\mathcal{J}_\pi = \begin{bmatrix} 1 & \bullet & \bullet & * & 0 \\ 0 & 1 & * & \bullet & 0 \\ 0 & 0 & 1 & \bullet & * \\ 0 & 0 & 0 & 1 & \bullet \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mu_\pi =$$

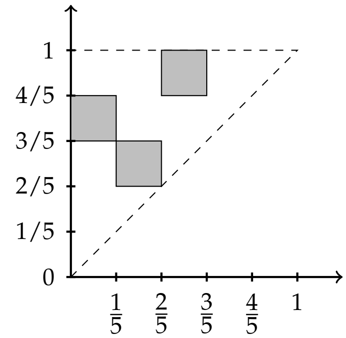


Figure 2: Set $\pi = \{\{1,4\},\{2,3,5\}\}$, we give an example of $\mathcal{J}_\pi$ (left picture) and μ_π (right picture). In $\mathcal{J}_\pi$ the symbol $*$ means that in that position there is an element of $\mathbb{K}^\times$, and $\bullet$ is an element of $\mathbb{K}$. In the graphic of μ_π we have that everywhere but the gray areas has zero weight, while the gray areas represent where the measure has uniform weight. Each square has total weight $\frac{1}{n} = \frac{1}{5}$, so that the total weight is $\int_\Delta d\mu = \frac{3}{5}$.

3.1 A combinatorial interpretation of the superplancherel measure

We associate to $\pi \vdash [n]$ the following set $\mathcal{J}_\pi \subseteq U_n$: a matrix A belongs to $\mathcal{J}_\pi$ iff

- if $(i,j) \in D(\pi)$ then $A_{i,j} \in \mathbb{K} \setminus \{0\}$;
- if $(i,j) \in \mathrm{Reg}(\pi) \setminus D(\pi)$ then $A_{i,j} = 0$;
- if $(i,j) \in \mathrm{Sing}(\pi)$ then $A_{i,j} \in \mathbb{K}$.

For an example of $\mathcal{J}_\pi$ see Figure 2. We want to stress out that the $\mathcal{J}_\pi$ are not the superclasses $\mathcal{K}_\pi$.

Lemma 3.1. *Given a matrix $A \in U_n$, there exists a unique π such that $A \in \mathcal{J}_\pi$. In other words, $U_n = \bigsqcup_{\pi \vdash [n]} \mathcal{J}_\pi$.*

The proof of this lemma is relatively easy and we will omit it in this extended abstract. It is clear that

$$|\mathcal{J}_\pi| = (q-1)^{d(\pi)} \cdot q^{|\mathrm{Sing}(\pi)|} = \frac{(q-1)^{d(\pi)} \cdot q^{2\dim(\pi) - 2d(\pi)}}{q^{\mathrm{crs}(\pi)}}.$$

Hence, we can see the superplancherel measure of π as the probability of choosing a random matrix in U_n which belongs to $\mathcal{J}_\pi$, that is, $\mathrm{SPl}_n(\chi^\pi) = \frac{|\mathcal{J}_\pi|}{|U_n|}$. We use this interpretation to generate the third picture of Section 1.

4 Set partitions as measures on the unit square

Set $\Delta = \{(x,y) \in [0,1]^2 \text{ s.t. } y \geq x\}$. In this section we describe an embedding of set partitions into particular measures on Δ. We settle first some notation: if $A \subseteq \mathbb{R}^2$ is measurable then we write λ_A for the uniform measure on A of total mass equal to 1, that is, $\int_A d\lambda_A = 1$; given $n \in \mathbb{N}, i < j \leq n$, set

$$A_{i,j} = \left\{(x,y) \in \mathbb{R}^2 \text{ s.t. } \frac{i-1}{n} \leq x \leq \frac{i}{n}, \frac{j-1}{n} \leq y \leq \frac{j}{n}\right\} \subseteq \Delta;$$

for $\pi \vdash [n]$ we will write $A_\pi := \bigcup_{(i,j)\in D(\pi)} A_{i,j}$ and $\mu_\pi = \frac{1}{n}\sum_{(i,j)\in D(\pi)} \lambda_{A_{i,j}}$. An example is given in the second part of Figure 2.

Definition 4.1. *Let $X \subseteq \mathbb{R}^2$, set π_1 (respectively π_2) the projection into the first (respectively the second) coordinate. A measure μ on X is said to have* uniform marginals *if for each interval $I \subseteq \pi_1(X)$ and $J \subseteq \pi_2(X)$*

$$\mu(I \times \pi_2(X)) = |I|, \qquad \mu(\pi_1(X) \times J) = |J|.$$

Similarly, the measure μ has subuniform marginals *if, for a pair of intervals $I \subseteq \pi_1(X)$ and $J \subseteq \pi_2(X)$,*

$$\mu(I \times \pi_2(X)) \leq |I|, \qquad \mu(\pi_1(X) \times J) \leq |J|.$$

As a measure on Δ, μ_π has subuniform marginals and in particular $\int_\Delta d\mu \leq 1$. We call *subprobability* a positive measure with total weight less than or equal to 1.

4.1 Statistics of set partitions approximated by integrals

We define the following space of measures:

$$\Gamma := \{\text{subprobabilities } \mu \text{ on } \Delta \text{ s.t. } \mu \text{ has subuniform marginals}\};$$

In this new setting we can describe the values of $d(\pi), \dim(\pi), \operatorname{crs}(\pi)$ as follows:

Lemma 4.2. *Let $\pi \vdash [n]$, so that $\mu_\pi \in \Gamma$, then*

1. $d(\pi) \in O(n)$;
2. $\dim(\pi) = n^2 \int_\Delta (y-x)\, d\mu_\pi(x,y)$;
3. $\operatorname{crs}(\pi) = n^2 \int_{\Delta^2} \mathbb{1}[x_1 < x_2 < y_1 < y_2]\, d\mu_\pi(x_1,y_1)\, d\mu_\pi(x_2,y_2) + O(n)$.

The proof of this lemma is technical and relatively uninteresting, hence we will not write it in this extended abstract (it is of course available in the original paper).

For each measure $\mu \in \Gamma$ we set

- $I_1(\mu) := \int_\Delta (y-x)\, d\mu$;
- $I_2(\mu) := \int_{\Delta^2} \mathbb{1}[x_1 < x_2 < y_1 < y_2]\, d\mu(x_1, y_1)\, d\mu(x_2, y_2)$;
- $I(\mu) := \frac{1}{2} - 2I_1(\mu) + I_2(\mu)$.

Hence for $\pi \vdash [n]$ we have $\mathrm{SPl}_n(\chi^\pi) = \exp\left(-n^2 \log q \cdot I(\mu_\pi) + O(n)\right)$.

4.2 Maximizing the entropy

We set

$$\tilde{\Gamma} := \{\text{subprobabilities } \mu \text{ on } [0,1/2] \times [1/2,1] \text{ s.t. } \mu \text{ has uniform marginals}\}.$$

Define the measure Ω with density $1/\sqrt{2}$ on the set $\left\{(x, 1-x) \text{ s.t. } x \in \left[0, \frac{1}{2}\right]\right\}$ and 0 elsewhere, and notice that $\Omega \in \tilde{\Gamma}$. The goal of this section is to prove the following proposition:

Proposition 4.3. *Consider $\mu \in \Gamma$, then $I(\mu) = 0$ if and only if $\mu = \Omega$.*

We will prove the proposition after studying the two functionals I_1 and I_2.

Lemma 4.4. *Let $\mu \in \Gamma$, then $I_1(\mu) = \int_\Delta (y-x)\, d\mu \leq 1/4$, with equality if and only if $\mu \in \tilde{\Gamma}$.*

Sketch of the proof. We "squeeze" the measure μ toward the upper left corner of Δ, in order to maximize I_1: consider the two functions $f_\mu(x) := \mu([0,x] \times [0,1]) \leq x$ and $g_\mu(y) := 1 - \mu([0,1] \times [y,1]) \geq y$, we define a subprobability $\tilde{\mu}$ as

$$\tilde{\mu}([0, f_\mu(x)] \times [g_\mu(y), 1]) := \mu([0,x] \times [y,1]),$$

and we claim that $I_1(\mu) \leq I_1(\tilde{\mu})$, with equality if and only if μ has uniform marginals. The unconvinced reader may look at the Figure 3 (first and second graphs) or at the extended version of this paper.

Call $l_\mu = \mu(\Delta)$, we notice that $\tilde{\mu}$ has uniform marginals, hence the x-marginal of $\tilde{\mu}$ is $\mathrm{Leb}([0, l_\mu])$, the Lebesgue measure on the interval $[0, l_\mu]$; similarly, the y-marginal of $\tilde{\mu}$ is $\mathrm{Leb}([1-l_\mu, 1])$. Hence

$$I_1(\tilde{\mu}) = \int_\Delta y\, d\tilde{\mu} - \int_\Delta x\, d\tilde{\mu} = \int_{1-l_\mu}^{1} y\, dy - \int_0^{l_\mu} x\, dx = l_\mu - l_\mu^2.$$

Since $l_\mu \leq 1$, the maximal value of $l_\mu(1-l_\mu)$ is obtained when $l_\mu = 1/2$, in which case $I_1(\tilde{\mu}) = 1/4$. Notice that if μ has uniform marginals and $l_\mu = 1/2$ then $\mu = \tilde{\mu} \in \tilde{\Gamma}$. $\square$

Lemma 4.5. *Let $\mu \in \tilde{\Gamma}$ such that $I_2(\mu) = 0$. Then $\mu = \Omega$.*

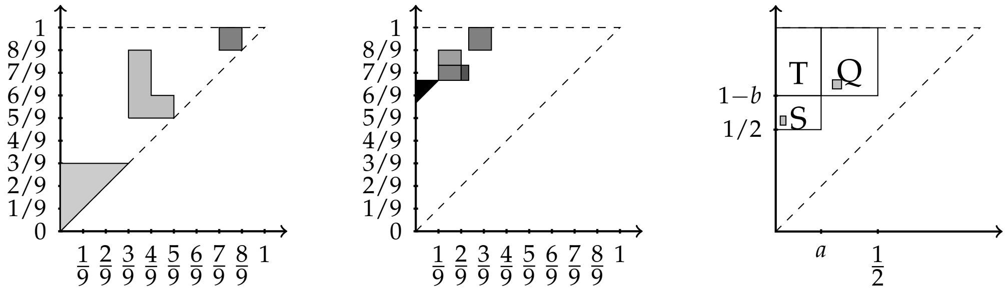


Figure 3: The first two graphics represent an example of the transformation of μ (left image) into $\tilde{\mu}$ (central image) of Lemma 4.4. The third graphic is an example of the area division in the proof of Lemma 4.5. If the measure μ has non zero weight inside S (here is pictured as the gray area), then it has also non zero weight in Q, and therefore $I_2(\mu) \neq 0$.

Sketch of the proof. Define F_ρ to be a variation of the distribution function for a measure $\rho \in \tilde{\Gamma}$: $F_\rho(a,b) := \rho([0,a] \times [1-b,1])$ for $a,b \in [0,1/2]$. To prove the lemma it is enough to show that $F_\mu(a,b) = F_\Omega(a,b) = \min(a,b)$. Suppose $a \leq b$ (the other case is done similarly), and consider the three sets $S = [0,a] \times [1/2, 1-b]$, $T = [0,a] \times [1-b,1]$, $Q = [a,1/2] \times [1-b,1]$ as in Figure 3. Because of the uniform marginals, one can prove that if $\int_S d\mu > 0$ then $\int_Q d\mu > 0$, so that

$$I_2(\mu) \geq \int_{S \times Q} \mathbb{1}[x_1 < x_2 < y_1 < y_2]\, d\mu(x_1,y_1)\, d\mu(x_1,y_2) = \mu(S) \cdot \mu(Q) > 0,$$

which is a contradiction. Thus $\int_S d\mu = 0$. This implies that $F_\mu(a,b) = \mu(T) = \mu(T \cup S) = a$ and the proof is concluded. □

Proof of Proposition 4.3. It is easy to see that $I(\Omega) = 0$. Suppose on the other hand that $I(\mu) = 0$, then $I_1(\mu) = \frac{1}{4} + \frac{I_2(\mu)}{2} \leq \frac{1}{4}$ by Lemma 4.4. This implies that $I_2(\mu) = 0$ and thus $I_1(\mu) = 1/4$; hence $\mu \in \tilde{\Gamma}$ by Lemma 4.4, and we can apply Lemma 4.5 to conclude that $\mu = \Omega$. □

5 Convergence in the weak* topology

In this section we sketch the proof of the main result of the paper, that is, that $\mu_{\pi^{(n)}}$ converges with the weak* convergence to Ω when $\pi^{(n)}$ is a random set partition distributed with the superplancherel measure. Recall that the weak* topology in the space of subprobabilities $M^{\leq 1}(\Delta)$ of the metric space $\Delta = \{(x, y) \in \mathbb{R}^2 \text{ s.t. } 0 \leq x \leq y \leq 1\}$ is defined as follows: for a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subseteq M^{\leq 1}(\Delta)$ and $\mu \in M^{\leq 1}(\Delta)$, then we say that $\mu_n \stackrel{w^*}{\to} \mu$ if $\int f(x)\,d\mu_n(x) \to \int f(x)\,d\mu(x)$ for each $f\colon \Delta \to \mathbb{R}$ continuous. It is known that since Δ is compact then there is a metric, called the *Lévy-Prokhorov distance* d_{L-P}, associated to the weak* convergence, in the sense that $\mu_n \stackrel{w^*}{\to} \mu$ if and only if $d_{L-P}(\mu_n, \mu) \to 0$.

Theorem 5.1. *We have*

$$\mathrm{SPl}(\{\pi \vdash [n] \textit{ s.t. } d_{L-P}(\mu, \Omega) > \epsilon\}) \to 0.$$

Proof. Due to space restrictions, we will not present here the proofs of the following 2 lemmas, which can nevertheless be found in the extended version of this paper:

- the space Γ is compact with respect to the weak* topology;
- the functional $I(\mu) := \frac{1}{2} - 2I_1(\mu) + I_2(\mu)$ is continuous with this topology.

We claim that for each $\epsilon > 0$ there exists $\delta > 0$ such that if $d_{L-P}(\mu, \Omega) > \epsilon$ then $|I(\mu)| > \delta$. Fix $\epsilon > 0$ and suppose the claim not true, so that for each $\delta > 0$ there is μ_δ with $d_{L-P}(\mu_\delta, \Omega) > \epsilon$ and $|I(\mu)| \leq \delta$. Set $\delta = 1/n$, we obtain a sequence (μ_n) with $|I(\mu_n)| \leq 1/n$. Since Γ is compact there exists a converging subsequence (μ_{i_n}). Call $\overline{\mu}$ the limit of this subsequence. Since I is continuous we have $I(\overline{\mu}) = \lim_n I(\mu_{i_n}) = 0$. This is a contradiction, since Ω is the unique measure in Γ with $I(\Omega) = 0$, and the claim is proved.

Fix $\epsilon > 0$, then there exists $\delta > 0$ such that if $d_{L-P}(\mu, \Omega) > \epsilon$ then $|I(\mu)| > \delta$. We define the set $N_\epsilon^n := \{\pi \vdash [n] \text{ s.t. } d_{L-P}(\mu, \Omega) > \epsilon\}$, then

$$\mathrm{SPl}(N_\epsilon^n) = \sum_{\pi \in N_\epsilon^n} \exp(-n^2 \log q I(\mu_\pi) + O(n)).$$

Recall that the number of set partitions of n, called the Bell number, is bounded from above by n^n; therefore

$$\mathrm{SPl}(N_\epsilon^n) \leq n^n \sup_{\pi \in N_\epsilon^n} \exp(-n^2 \log q I(\mu_\pi) + O(n)) < \exp(-n^2 \delta \log q + O(n \log n)) \to 0. \quad \square$$

Corollary 5.2. *For each $n \geq 1$ let π_n be a random set partition of n distributed with the superplancherel measure SPl_n, then*

$$\mu_{\pi_n} \to \Omega, \quad \frac{\dim(\pi_n)}{n^2} \to \frac{1}{4}, \quad \mathrm{crs}(\pi_n) \in o(n^2) \qquad \textit{almost surely.}$$

Proof. We prove only the convergence of μ_{π_n}, since the cases $\dim(\pi_n)$ and $\mathrm{crs}(\pi_n)$ are similar. As before, set $N_\epsilon^n := \{\pi \vdash [n] \text{ s.t. } d_{L-P}(\mu, \Omega) > \epsilon\}$, so that the superplancherel measure is bounded: $\mathrm{SPl}(N_\epsilon^n) < \exp(-n^2 \delta \log q + O(n \log n))$. Thus $\sum_n \mathrm{SPl}_n(N_n^\epsilon) < \infty$ and we can apply the first Borel Cantelli lemma, which implies that $\limsup_n N_n^\epsilon$ has measure zero for each $\epsilon > 0$, and therefore $\mu_{\pi_n} \to \Omega$ almost surely. $\square$

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